Algebras of functions. The functor

$$\mathbf{Set}(-,\mathbb{C}) : \mathbf{Set} \longrightarrow \mathbf{Alg}^{op}$$

is defined as follows. The multiplication and the unit of $\mathbf{Set}(X,\mathbb{C})$ for any set X we construct from the comonoid structure of X.

$$\mathbf{Set}(X,\mathbb{C}) \otimes \mathbf{Set}(X,\mathbb{C}) \to \mathbf{Set}(X \times X, \mathbb{C}) \xrightarrow{\mathbf{Set}(\Delta,\mathbb{C})} \mathbf{Set}(X,\mathbb{C})$$

$$f_1 \otimes f_2 \longmapsto \big((x_1,x_2) \mapsto f_1(x_1) f_2(x_2)\big) \mapsto \big(x \mapsto f_1(x) f_2(x)\big),$$

(element-wise multiplication of values)

$$\mathbb{C} \xrightarrow{\cong} \mathbf{Set}(\bullet, \mathbb{C}) \xrightarrow{\mathbf{Set}(\varepsilon, \mathbb{C})} \mathbf{Set}(X, \mathbb{C})$$

(constant unit).

$\mathbf{Set}(-, \mathbb{C})$ transforms coassociativity and counitality into associativity and unitality.

Proposition 5. The function algebra functor $\mathbf{Set}(-, \mathbb{C})$ extends canonically from $\mathbf{Set}$ to $\mathbf{Coalg}$ as follows

$$
\begin{tikzcd}
\mathbf{Set} \arrow[r, hook, "\mathbb{C}-"] \arrow[d, "\mathbf{Set}(-{,}\mathbb{C})"'] & \mathbf{Coalg} \arrow[dl, dotted, "\mathbf{Vect}(-{,}\mathbb{C})"] \\
\mathbf{Alg}^{op} &
\end{tikzcd}
\qquad (*)
$$

Proof. The functor

$$\mathbf{Vect}(-,\mathbb{C}) : \mathbf{Coalg} \longrightarrow \mathbf{Alg}^{op}$$

is defined as follows. The multiplication and the unit of $\mathbf{Vect}(C,\mathbb{C})$ for any coalgebra C we construct from the comonoid structure of C.

$$\mathbf{Vect}(C,\mathbb{C})\otimes\mathbf{Vect}(C,\mathbb{C})\longrightarrow\mathbf{Vect}(C\otimes C,\mathbb{C})\xrightarrow{\mathbf{Vect}(\Delta,\mathbb{C})}\mathbf{Vect}(C,\mathbb{C})$$

$$f_1\otimes f_2\longmapsto\Big(c_1\otimes c_2\longmapsto f_1(c_1)f_2(c_2)\Big)\longmapsto\big(c\mapsto f_1(c_{(1)})f_2(c_{(2)})\big),$$

$$\mathbb{C}\longrightarrow\mathbf{Vect}(\mathbb{C},\mathbb{C})\xrightarrow{\mathbf{Vect}(\varepsilon,\mathbb{C})}\mathbf{Vect}(C,\mathbb{C})$$

$\mathbf{Vect}(-,\mathbb{C})$ transforms coassociativity and counitality into associativity and unitality.

By Proposition 2 the bijection natural in X

$$\mathbf{Set}(X,\mathbb{C}) \xrightarrow{\cong} \mathbf{Vect}(\mathbb{C}X,\mathbb{C})$$

satisfies commutativity of the diagrams

$$\begin{tikzcd}
\mathbf{Set}(X,\mathbb{C})\otimes\mathbf{Set}(X,\mathbb{C}) \arrow[r] \arrow[d, "\cong"'] & \mathbf{Set}(X\times X,\mathbb{C}) \arrow[r, "\mathbf{Set}(\Delta{,}\mathbb{C})"] \arrow[d, "\cong"'] & \mathbf{Set}(X,\mathbb{C}) \arrow[d, "\cong"'] \\
\mathbf{Vect}(\mathbb{C}X,\mathbb{C})\otimes\mathbf{Vect}(\mathbb{C}X,\mathbb{C}) \arrow[r] & \mathbf{Vect}(\mathbb{C}X\otimes\mathbb{C}X,\mathbb{C}) \arrow[r] & \mathbf{Vect}(\mathbb{C}X,\mathbb{C})
\end{tikzcd}$$

(in the middle we used also the inverse isomorphism

$\mathbb{C}X\otimes\mathbb{C}X \xrightarrow{\cong} \mathbb{C}(X\times X)$ in $\mathbf{Vect}$, natural in X)

$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, equal] & \mathbf{Set}(\bullet, \mathbb{C}) \arrow[rr, "\mathbf{Set}(\varepsilon, \mathbb{C})"] \arrow[d, "\cong"'] & & \mathbf{Set}(X, \mathbb{C}) \arrow[d, "\cong"'] \\
\mathbb{C} \arrow[r] & \mathbf{Vect}(\mathbb{C}, \mathbb{C}) \arrow[rr, "\mathbf{Vect}(\varepsilon, \mathbb{C})"] & & \mathbf{Vect}(\mathbb{C}X, \mathbb{C})
\end{tikzcd}
$$

(in the middle we used also the inverse isomorphism

$\mathbb{C} \xrightarrow{\cong} \mathbb{C}\bullet$ in $\mathbf{Vect}$) which implies that

the diagram $(*)$ commutes. $\blacksquare$

Excercise 1. Check coalgebra axioms for Examples 1 and 2 and find algebra presentations (generators and relations) of their dual algebras, assuming in Example 1 that the category $\mathcal{C}$ is finite.

Excercise 2. Show that the algebra $\mathbb{C}[[x]]$ of formal power series in one variable is dual to some coalgebra. Is the algebra $\mathbb{C}[x]$ of polynomials dual to any coalgebra?

<u>Solution.</u> $\mathbb{C}[[x]] = \lim\limits_{n} \mathbb{C}[x]/(x^{n+1}) = \lim\limits_{n} \left(\mathbb{C}[x]/(x^{n+1})\right)^{**}$

$= \left(\operatorname{colim}\limits_{n} \left(\mathbb{C}[x]/(x^{n+1})\right)^{*}\right)^{*}$

Assume $\mathbb{C}[x] = \mathbf{Vect}(C, \mathbb{C})$

$C = \operatorname{colim}_i C_i$, $\dim C_i < \infty$, C_i subcoalgebra

$\mathbb{C}[x] = \mathbf{Vect}(C, \mathbb{C}) = \mathbf{Vect}(\operatorname{colim}_i C_i, \mathbb{C}) = \lim_i \mathbf{Vect}(C_i, \mathbb{C})$

$= \lim_i A_i$, $\dim A_i < \infty$.

$\mathbb{C}[x] \xrightarrow{\varphi_i} A_i$ $\quad \varphi_i(A_i) = \bar{A}_i$, $\dim \bar{A}_i < \infty$

$\mathbb{C}[x] \overset{\bar{\varphi}_i}{\twoheadrightarrow} \bar{A}_i$, $\quad \mathbb{C}[x] = \lim_i \bar{A}_i$

But $\mathbb{C}$ alg. closed $\Rightarrow$ every $\bar{A}_i = \mathbb{C}[x] / ((x - x_{i,1}) \cdots (x - x_{i,n_i}))$

$= \prod_i \mathbb{C}[x] / ((x - x_i)^{d_i})$

$\Rightarrow \varprojlim_i \overline{A}_i$ = product of completions $\mathbb{C}[[x-x_i]]$ of $\mathbb{C}[x]$ with respect of the maximal ideals $(x-x_i)$ and finite dimensional algebras $\mathbb{C}[x]/(x-x_i)^{d_i}$.

If it is without nontrivial nilpotents and idempotents then it must be $\mathbb{C}[[x-x_{i_0}]]$.

Now, $\mathbb{C}[x] = \mathbb{C}[x-x_{i_0}] \hookrightarrow \mathbb{C}[[x-x_{i_0}]]$ is injective but $(1-(x-x_{i_0}))^{-1} = 1 + (x-x_{i_0}) + (x-x_{i_0})^2 + \dots \in \mathbb{C}[[x-x_{i_0}]]$ is not a polynomial. Therefore $\mathbb{C}[x]$ cannot be dual to any coalgebra.

Another argument: $\dim \mathbb{C}[x] = \aleph_0$. If $\mathbb{C}[x] = C^*$ for some C then $\dim C$ infinite $\Rightarrow$ $\dim C \geqslant \aleph_0$. Assume X is a basis of C. Then $C^* = \mathbf{Vect}(C, \mathbb{C}) = \mathbf{Set}(X, \mathbb{C})$. For any finite family of independent subsets $S \subset X$ the set of characteristic functions χ_S is linearly independent $\Rightarrow$ $\dim C^* \geqslant 2^{\aleph_0} > \aleph_0 = \dim \mathbb{C}[x]$ $\lightning$

$S_1, \dots, S_n$ independent subsets means that

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Excercise 3. Let X be a set of powers $\partial^k \in \mathbb{C}[\partial]$.
and $C = \mathbb{C}X$ with

$$\Delta(\partial^k) = \sum_{i=0}^{k} \binom{k}{i} \partial^i \otimes \partial^{k-i}, \quad \varepsilon(\partial^k) = \begin{cases} 1 & k=0 \\ 0 & k>0. \end{cases}$$

Check coalgebra axioms and identify the dual algebra.
Explain the relation with the Leibniz rule and the Taylor expansion.

Excercise 4. For any algebra A consider

$$A^\circ := \{ f \in A^* \mid f \text{ factors through a finite dim. algebra } F \}$$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, dashed, "\mathbf{Alg}"'] & & \mathbb{C} \\
& F \arrow[ur, dashed, "\mathbf{Vect}"'] &
\end{tikzcd}$$

Show that $A^\circ = \{ \alpha \in A^* \mid \exists \underset{\operatorname{codim} I < \infty}{I \lhd A} \ \alpha(I) = 0 \}$.

<u>Solution</u>

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\varphi"'] & & \mathbb{C} \\ & F \arrow[ur, "\lambda"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} & A \arrow[rr, "\alpha"] \arrow[dl, "\pi"'] \arrow[dr] & & \mathbb{C} \\ \varphi(A) \arrow[rr, hook, "\iota"'] & & F \arrow[ur, "\lambda"'] & \end{tikzcd} \Longrightarrow \begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & \mathbb{C} \\ & A/I = \varphi(A) \arrow[ur, "\lambda\circ\iota"'] & \end{tikzcd}$$

$$\Downarrow$$

$$\alpha(I) = 0$$

$$\operatorname{codim} I < \infty \quad ,$$

$\exists\, I \lhd A$ $\quad \alpha(I) = 0 \Rightarrow$
$\operatorname{codim} I < \infty$

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr] & & \mathbb{C} \\ & A/I \arrow[ur, dotted, "\exists!\,\bar{\lambda}"'] & \end{tikzcd}$$

A/I ← fin. dim.

Proposition 5. The algebra structure on A induces a coalgebra structure on A° and an algebra map

$$A \to \mathbf{Vect}(A^\circ, \mathbb{C}),$$

both natural in A.

Proof. We need three lemmas.

Lemma 1.
$\varphi \in \mathbf{Alg}(A,B) \Rightarrow \varphi^*: B^* \to A^*$ satisfies $\varphi^*(B^\circ) \subset A^\circ$.

Proof. $J \triangleleft B$, $\operatorname{codim} J < \infty$

$$0 \to \varphi^{-1}(J) \to A \to B/J \quad \text{exact}$$

$\Rightarrow \operatorname{codim} \varphi^{-1}(J) < \infty$.

$\beta \in B^\circ$, $\beta(J) = 0 \Rightarrow \varphi^*(\beta)(\varphi^{-1}(J)) = \beta(J) = 0$

$\Rightarrow \varphi^*(\beta) \in A^\circ$. $\blacksquare$

Lemma 2. Under the embedding $A^* \otimes B^* \hookrightarrow (A \otimes B)^*$

$$A^\circ \otimes B^\circ = (A \otimes B)^\circ .$$

Proof. $K \lhd (A \otimes B)$, $\operatorname{codim} K < \infty$, $A \xrightarrow{\varphi} A \otimes B$, $a \mapsto a \otimes 1$

$$I := \{a \in A \mid a \otimes 1 \in K\} = \varphi^{-1}(K)$$

$$\overset{\text{Lemma 1}}{\Longrightarrow} \operatorname{codim} I < \infty$$

Similarly for $J := \{b \in A \mid 1 \otimes b \in K\}$.

$A \otimes J + I \otimes B \subset K$ and $A \otimes J + I \otimes B = \ker\left(A \otimes B \to A/I \otimes B/J\right)$

fin. dim.

$$\Rightarrow \operatorname{codim}(A \otimes J + I \otimes B) < \infty .$$

$\gamma \in (A\otimes B)^\circ$, $\quad \gamma(K)=0 \implies \gamma(A\otimes J + I\otimes B) = 0$

$$\begin{tikzcd}
A\otimes B \arrow[rr, "\gamma"] \arrow[dr, "\varpi"'] & & \mathbb{C} \\
& A/I \otimes B/J \arrow[ur, dashed, "\exists!\lambda"'] &
\end{tikzcd}$$

both fin. dim

$\implies \lambda \in (A/I \otimes B/J)^* \cong (A/I)^* \otimes (B/J)^*$

$$\lambda = \sum_i \alpha_i \otimes \beta_i$$

$$\lambda(a\otimes b) = \sum_i \alpha_i(a)\beta_i(b)$$

$A \xrightarrow{\pi} A/I$, $\quad B \xrightarrow{\rho} B/J \quad \implies \quad \alpha_i\circ\pi \in A^\circ$, $\beta_i\circ\rho \in B^\circ$

$\implies \gamma = \lambda\circ\varpi = \sum_i (\alpha_i\circ\pi)\otimes(\beta_i\circ\rho) \in A^\circ\otimes B^\circ \implies (A\otimes B)^\circ \subset A^\circ\otimes B^\circ$.

To prove $A^\circ \otimes B^\circ \subset (A\otimes B)^\circ$ we take $\alpha \in A^\circ$, $\beta \in B^\circ$

$\alpha(I) = 0$, $\operatorname{codim} I < \infty$, $\beta(J) = 0$, $\operatorname{codim} J < \infty$

$\Rightarrow$ $(\alpha\otimes\beta)(A\otimes J + I\otimes B) = 0$, $\operatorname{codim}(A\otimes J + I\otimes B) < \infty$

$\Rightarrow$ $\alpha\otimes\beta \in (A\otimes B)^\circ$. $\square$

Lemma 3. For the multiplication map $m: A\otimes A \to A$

$$m^*(A^\circ) \subset A^\circ\otimes A^\circ.$$

Proof. $\alpha \in A^\circ$, $a_1, a_2 \in A$ $\Rightarrow$ $m^*(\alpha)(a_1\otimes a_2) = \alpha(a_1 a_2)$.

$\alpha(I) = 0$, $\operatorname{codim} I < \infty$

$\Rightarrow$ $m^*(\alpha)(A\otimes I + I\otimes A) = 0$, $\operatorname{codim}(A\otimes I + I\otimes A) < \infty$

$\Rightarrow$ $m^*(\alpha) \in (A\otimes A)^\circ \underset{\text{Lemma 2}}{=} A^\circ\otimes A^\circ$. $\square$

Therefore we can define

$$\Delta : A^\circ \longrightarrow A^\circ \otimes A \quad , \qquad \Delta(\alpha) := m^*(\alpha)$$

$$\varepsilon : A^\circ \longrightarrow \mathbb{C} \quad , \qquad \varepsilon(\alpha) := \alpha(1) .$$

Consider the diagram

$$\begin{tikzcd}
 & & (A\otimes A)^* \arrow[rrd, "(A\otimes m)^*"] & & \\
A^* \arrow[rru, "m^*"] \arrow[dr, "m^*"] & & A^\circ\otimes A^\circ \arrow[u, hook] \arrow[rrdd, "A^\circ\otimes\Delta"] & & (A\otimes A\otimes A)^* \\
 & (A\otimes A)^* \arrow[rrru, "(m\otimes A)^*"] & & & \\
A^\circ \arrow[uu, hook] \arrow[rruu, "\Delta"] \arrow[dr, "\Delta"'] & & & & A^\circ\otimes A^\circ\otimes A^\circ \arrow[uu, hook] \\
 & A^\circ\otimes A^\circ \arrow[uu, hook] \arrow[rrru, "\Delta\otimes A^\circ"'] & & &
\end{tikzcd}$$

.

The front square commutes by definition of Δ.
The rear squares commute by definition of Δ and $(A \otimes m)^* = A^* \otimes m^*$.
Similarly the front and rear rectangles commute.
The top rectangle commutes by associativity of comultiplication and functoriality of dualization.
Therefore, since the vertical maps are injective, the bottom rectangle commutes as well, which proves coassociativity.
Counitality follows from the definition of ε, unitality of A and functoriality of dualization.

The map $A \to \mathbf{Vect}(A^\circ, \mathbb{C})$ is

$$a \longmapsto (\alpha \longmapsto \alpha(a)) .$$

Everything is obviously natural in A. $\square$

Remark. The kernel of this algebra map contains the intersection of all finite codimension ideals. Up to this kernel it is <u>generalized functional representation of A</u> as generalized functions on a generalized set A°.